Liquids in multi-orbital SU(N) Magnets with Ultracold Alkaline Earth Atoms

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In this work we study one family of liquid states of $k-$orbital SU(N) spin systems, focusing on the case of $k = 2$ which can be realized by ultracold alkaline earth atoms trapped in optical lattices, with N as large as 10. Five different algebraic liquid states with selectively coupled charge, spin and orbital quantum fluctuations are considered. The algebraic liquid states can be stabilized with large enough N, and the scaling dimension of physical order parameters is calculated using a systematic $1/N$ expansion. The phase transitions between these liquid states are also studied, and all the algebraic liquid states discussed in this work can be obtained from one "mother" state with SU(2) $\times$ U(1) gauge symmetry.

I. INTRODUCTION

Spin liquid state as an exotic quantum ground state of strongly correlated systems has been studied for decades[1,2]. Thanks to the active search for spin liquids in materials in the last few years[3,4], people are encouraged to believe in the existence of spin liquids in nature. The stability of spin liquid usually relies on large number of matter fields which suppress the continuous gauge field fluctuations. For instance, in the famous organic salts $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$[3,4], one of the proposed candidate spin liquid involves a spinon fermi surface, where the finite density of states of matter field tends to suppress the U(1) gauge field[5,6]. When the spinon fermi sea shrinks to a Dirac point, one needs to introduce large enough flavor number (N_f) of Dirac fermions to stabilize the spin liquid. However, large N_f is difficult to realize in SU(2) spin system, therefore one is motivated to look for systems with large spin symmetries. Tremendous theoretical and numerical efforts were made on SU(N) and Sp(N) spin systems with large N[7–16].

It was proposed that spin-3/2 cold atoms can realize Sp(4) symmetry without fine-tuning[17]. Recently it has been discovered that an exact SU(N) spin symmetry with N as large as 10 can be realized with alkaline earth cold atoms without fine-tuning any parameter[18]. Because the electrons carry zero total angular momentum, all the spin components belong to nuclear spins and hence the interaction between atoms are totally independent of the spin components, *i.e.* the system has SU(N) symmetry with $N = 2S + 1$ for nuclear spin-S. Therefore the alkaline earth cold atom plus optical lattice is a very promising system to realize the long-sought spin liquids. Besides the SU(N) spins, there is another orbital degree of freedom associated with the alkaline earth atoms, because both the ${}^1S_0$ and ${}^3P_0$ orbital levels (denoted as g and e respectively) have SU(N) spin symmetry[18].

Most generally this system has symmetry SU$(N)_s \times$ U$(1)_c \times$ U$(1)_o$. U$(1)_c$ corresponds to the conservation of the total atom number *i.e.* the charge U(1) symmetry; U$(1)_o$ corresponds to the conservation of $n_e - n_g$ *i.e.* the orbital U(1) symmetry. We will tentatively assume the system has an extra orbital Z_2 symmetry corresponding to switching e and g *i.e.* $\exp(i\frac{\pi}{2}\sigma^x)$, therefore we take the hopping amplitude of these two orbitals to be equal, also the two intraorbital Hubbard interactions are equal. Weak violation of this Z_2 symmetry will be discussed in this paper, and we will show that it is irrelevant to the main physics discussed in this paper. Under these assumptions, after straightforward algebraic calculations the Hamiltonian in Eq. 2 of Ref.[18] can be rewritten as

$$H = \sum_{\langle i,j\rangle\alpha,m} -tc^\dagger_{i\alpha m}c_{j\alpha m} + H.c. + \sum_i U(n_i - \bar{n})^2 + \sum_a J(T^a_i)^2 + J_z(T^z_i)^2. \quad (1)$$

$m = 1\cdots N$, and $\alpha = e,\ g$. Here $n_i = \sum_{\alpha m} n_{i\alpha m}$ is the total number of the atoms on each site, $T^a_i = c^\dagger_{i\alpha m}\sigma^a_{\alpha\beta}c_{i\beta m}$ is the pseudospin vector of orbital levels. U, J and J_z are simple linear combinations between U, V_{ex} and V in Ref.[18], which are from the $s-$wave scatterings between atoms. Since different orbital channels have different scattering lengths, J and J_z terms are allowed to exist because otherwise the system will have an unphysical SU($2N$) symmetry. Eq. 1 is the starting point of our study, and since all the fermionic alkaline atoms under study carry half integer nuclear spins, N will be taken to be even hereafter.

In order to obtain more solid and quantitative results, we will keep both orbitals of the atoms at half-filling *i.e.* $\bar{n} = N$, $\sum_i n_{i,e} - n_{i,g} = 0$ and put this model on a honeycomb lattice with only nearest neighbor hopping. Therefore on top of the global symmetries discussed before, there is another particle-hole symmetry with $c_{j\alpha m} \to \eta_j c^\dagger_{j\alpha m}$, and $\eta_j = 1$ and -1 with j belonging to sublattices A and B respectively. If t is the dominant energy scale of the Hamiltonian, the half-filled fermions on honeycomb lattice is a semimetal with two Dirac valleys in the momentum space located at $\vec{Q} = (\pm\frac{4\pi}{3}, 0)$, and at low energy the band structure can be described by the Dirac Lagrangian

$$L = \sum_{a=1}^{4N} \bar{\psi}_a\gamma_\mu\partial_\mu\psi_a, \quad (\gamma_0, \gamma_1, \gamma_2) = (\tau^z, \tau^y, -\tau^x). \quad (2)$$

The 2$\times$2 Dirac matrices τ^i are operating on the two sites

in each unit cell on the honeycomb lattice. The Dirac fermion has two Dirac points at the corners of the Brillouin zone, therefore there are in total $N_f = 4N$ flavors of 2-component Dirac fermions, with an enlarged O($8N$) flavor symmetry at low energy, which will be manifest after we rewrite the Dirac fermions in terms of Majorana fermions. The short-range interactions between the Dirac fermions are irrelevant at the free Dirac fermion fixed point.

In the following we will mostly be focusing on the Mott Insulator phase of Eq. 1 with U dominant. Motivated by the spin liquid and weak Mott insulator $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$[4,19], we want the system close to the Mott transition so that at short distance it still behaves like a semimetal, while at long distance the electron $c_{i\alpha m}$ fractionalizes. With $J = J_z = 0$, the existence of a fractionalized phase close to the Mott transition on the honeycomb lattice was shown with a slave rotor calculation in Ref.[19], and the fractionalized spinon has the same meanfield band structure as the Dirac semimetal. In our system with nonzero J and J_z, various strongly correlated liquid states with coupled spin, charge and orbital fluctuations can be realized in different parameter regimes of Eq. 1, and all the liquid states can be obtained from the $\mathrm{U}(1)\times\mathrm{SU}(2)$ spin liquid that will be studied first.

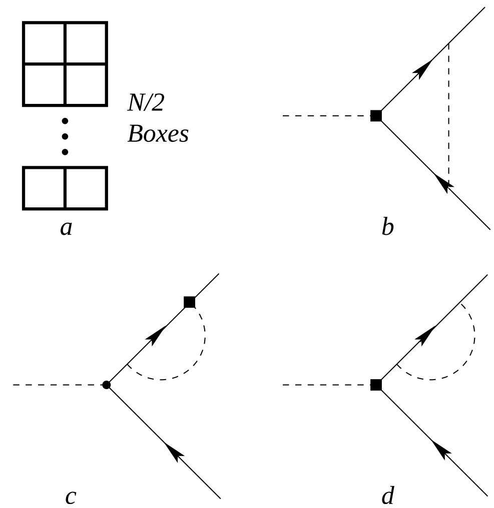

FIG. 1: a, the Young tableau of the representation of N SU(N) fermions on each site when orbital is constrained to be $\mathrm{SU}(2)_o$ singlet, N has to be an even number. b, c and d, Feynman diagrams which contribute to the RG flow of the velocity anisotropy Eq. 13, the solid square stands for the vertex $\sigma^3\gamma_k\partial_k$.

II. LIQUID STATES

A. U(1) × SU(2) spin liquid, the mother state

As the first example of liquid state, let us take both U, J dominate t, while keeping $J_z = 0$ tentatively. In this case the symmetry of Eq. 1 is enhanced to $\mathrm{SU}(N)_s\times\mathrm{U}(1)_c\times\mathrm{SU}(2)_o$. When U and J both dominate the kinetic energy, the system forbids charge fluctuations away from half-filling $n = N$ on each site, and also forbids orbital-triplet fluctuations, *i.e.* the low energy subspace of the Hilbert space only contains orbital $\mathrm{SU}(2)_o$ singlet. The Young tableau of the SU(N) representation on each site has two columns with $N/2$ boxes each column, which is a large-N generalization of SU(2) spin-1 (Fig. 1a). The half-filling constraint on the low energy Hilbert space implies that one can do a local U(1) rotation on the fermions, which will be manifested by introducing a U(1) gauge field a_μ coupled to the charge degree of freedom as usual. The orbital-singlet constraint on each site implies that the local $\mathrm{SU}(2)_o$ transformation will not change the physical state, and this local invariance can be described by a SU(2) gauge field coupled to the orbital indices of $c_{i\alpha m}$.

More formally, one can introduce the bosonic U(1) slave rotor b_i and SU(2) slave rotor, 2×2 matrix field $h_{\alpha\beta}$, as well as fermionic spinon $f_{i\alpha m}$ as following[19]:

$$c_{i\alpha m} = b_i h_{\alpha\beta} f_{i\beta m}. \tag{3}$$

We will call b the chargeon and $h_{\alpha\beta}$ the triplon field. h is a group element of SU(2), with $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ transformation: $h \to \mathcal{M}_L h \mathcal{M}_R$. The $\mathrm{SU}(2)_L$ symmetry is the physical $\mathrm{SU}(2)_o$ symmetry of the orbitals, while the $\mathrm{SU}(2)_R$ symmetry is a local SU(2) gauge symmetry, which leaves the physical operator $c_{i\alpha m}$ invariant with an accompanied SU(2) gauge transformation on $f_{i\alpha m}$: $f \to \mathcal{M}_R^{-1} f$. The chargeon b_i grants the spinon $f_{i\alpha m}$ a U(1) gauge symmetry as usual $b_i \to b_i e^{i\theta_i}$, $f_{i\alpha m} \to f_{i\alpha m} e^{-i\theta_i}$, and b_i also carries the $\mathrm{U}(1)_c$ charge *i.e.* b_i will couple to the external electromagnetic field if the fermions $c_{i\alpha m}$ were electrons. The properties of U(1) and SU(2) slave rotors were discussed in Ref.[19] and Ref.[20] respectively, although the SU(2) slave rotors in Ref.[20] was engineered from a very different set-up.

The U(1) and SU(2) gauge symmetry can be manifested by reformulating the hopping term of Eq. 1 using the decomposition of fermion operator Eq. 3:

$$H = \sum_{<i,j>} -t b_i^\dagger b_j f_{i\alpha}^\dagger h_{i\alpha\rho}^\dagger h_{j\rho\beta} f_{j\beta} + H.c. \tag{4}$$

And spinon $f_{i\alpha m}$ hops effectively in a band structure described by the following meanfield Hamiltonian

$$\begin{aligned} H &= \sum_{<i,j>} -t\langle U_{ij,\alpha\beta}\rangle f_{i\alpha}^\dagger f_{j\beta} + H.c. \\ \langle U_{ij,\alpha\beta}\rangle &= \langle b_i^\dagger b_j h_{i\alpha\rho}^\dagger h_{j\rho\beta}\rangle \end{aligned} \tag{5}$$

The value of $\langle U_{ij,\alpha\beta}\rangle$ should be solved self-consistently. If the self-consistent solution $\langle U_{ij,\alpha\beta}\rangle \sim \delta_{\alpha\beta}$, the U(1) and SU(2) symmetries are preserved by this meanfield solution. And the fluctuation on the meanfield solution is the gauge fields: $U_{ij,\alpha\beta} \sim \langle U_{ij,\alpha\beta}\rangle e^{-ia_{ij}-\sum_{l=1}^{3} iA_{ij}^l\tau^l/2}$. The dynamics of slave rotor b and $h_{\alpha\beta}$ are given by the

meanfield decompositions $-tb_i^\dagger b_j\langle f_{i\alpha}^\dagger h_{i\alpha\rho}^\dagger h_{j\rho\beta}f_{j\beta}\rangle$ and $-th_{i\alpha\rho}^\dagger h_{j\rho\beta}\langle b_i^\dagger b_j f_{i\alpha}^\dagger f_{j\beta}\rangle$ respectively.

In the Mott Insulator phase but close to the Mott transition, the spin model after second order t/U perturbation will be very complicated. However, there is another self-consistent way of studying this system. Motivated by the existence of the spinon fermi sea of weak Mott insulator $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$, we assume here the weak Mott insulator phase is a phase in which the chargeon b_i and triplon h are both gapped, and the fermionic spinon $f_{i\alpha m}$ fills the same mean field band structure as the original fermions $c_{i\alpha m}$ in the semimetal phase with $N_f = 4N$ flavors of 2-component Dirac fermions at low energy (Eq. 2), and then we can check the stability of this state. This spin liquid state corresponds to a mean field solution $\langle b_i^\dagger b_j h_{i\alpha\rho}^\dagger h_{j\rho\beta}\rangle = U\delta_{\alpha\beta}$, which preserves the U(1) and SU(2) gauge symmetry. After taking into account of the U(1) and SU(2) gauge fluctuation, the low energy field theory of this spin liquid is described by the following 2+1d electro-weak theory like Lagrangian:

$$\mathcal{L}_{ew} = \sum_{a=1}^{2N} \bar{\psi}_a\gamma_\mu(\partial_\mu - ia_\mu - \sum_{l=1}^{3} iA_\mu^l\frac{\sigma^l}{2})\psi_a + \cdots \quad (6)$$

Here $(\gamma_0,\gamma_1,\gamma_2) = (\tau^z,\tau^y,-\tau^x)$. ψ is the low energy mode of spinon f, and $\psi_1 = e^{i\frac{4\pi}{3}x}f$, $\psi_2 = e^{-i\frac{4\pi}{3}x}i\tau^y f$. Unlike Eq. 2, in Eq. 6 each Dirac fermion ψ is a four component fermion, because it contains both the Dirac indices and SU(2) gauge indices.

The global symmetry of Eq. 6 is SU($2N$), which is a combined symmetry of SU(N) spin symmetry and Dirac valley rotation. ψ transforms nontrivially under translation, space reflection, rotation, time reversal, and particle-hole transformation as following:

$$\begin{aligned}
\mathrm{Tr}_1 &: x\to x+1, \psi\to e^{i\frac{4\pi}{3}\mu^z}\psi,\\
\mathrm{Tr}_2 &: x\to x+\frac{1}{2}, y\to y+\frac{\sqrt{3}}{2}, \psi\to e^{i\frac{2\pi}{3}\mu^z}\psi,\\
\mathrm{T} &: t\to -t,\ \psi\to\gamma^1\mu^y\psi,\ a_\mu,\to -a_\mu,\\
&\quad A_\mu^1, A_\mu^3\to -A_\mu^1, -A_\mu^3,\\
\mathrm{P}_{\bar{a},x} &: x\to -x,\ \psi\to\gamma^1(\vec{r}_{\bar{a}}\cdot\vec{\mu})\psi,\ a_1, A_1^l\to -a_1, -A_1^l,\\
\mathrm{P}_y &: y\to -y,\ \psi\to\gamma^2\mu^z\psi,\ a_2, A_2^l\to -a_2, -A_2^l,\\
\mathrm{PH} &: c_{j\alpha m}\to\eta_j c_{j\alpha m}^\dagger,\ \psi\to\gamma^2\mu^x\psi^\dagger, a_\mu\to -a_\mu,\\
&\quad A_\mu^1, A_\mu^3\to -A_\mu^1, -A_\mu^3,\\
\mathrm{R}_{2\pi/3} &: \psi\to e^{i\frac{2\pi}{3}\gamma^0}\psi. \qquad (7)
\end{aligned}$$

μ^a are three Pauli matrices that operate on the two Dirac valleys. Notice that the hexagons of the triangular lattice form a triangular lattice with three sublattices, and $P_{\bar{a},x}$ is the reflection centered at sublattice $\bar{a}$ of the three sublattices. Vectors $\vec{r}_1 = (0,1)$, $\vec{r}_2 = (\frac{\sqrt{3}}{2},-\frac{1}{2})$, $\vec{r}_3 = (-\frac{\sqrt{3}}{2},-\frac{1}{2})$. In the equation above, transformations of gauge field components are not shown unless they transform nontrivially. $\mathrm{R}_{2\pi/3}$ is the hexagon centered rotation by $2\pi/3$. Notice that time reversal transformation (T) always comes with a complex conjugate transformation, and hence T only changes the sign of the SU(N) as well as SU(2)$_o$ generators that are antisymmetric and purely imaginary, therefore the SU(N) and SU(2)$_o$ Lie algebras are preserved.

The gauge symmetry and global symmetry together guarantee that none of the apparently relevant perturbations like fermion bilinears exists in the Lagrangian Eq. 6. When N is large enough the Lagrangian in Eq. 6 is a conformal field theory (CFT). The ellipses in Eq. 6 include all the gauge invariant four fermion interaction terms which break the SU($2N$) global symmetry down to the symmetries of the microscopic Hamiltonian Eq. 1. All these four fermion interactions are irrelevant for large enough N. This CFT fixed point is a pure spin liquid state because both the charge and orbital fluctuations are forbidden. The scaling dimension of gauge invariant physical order parameters at this CFT fixed point can be calculated using a systematic $1/N$ expansion in a similar way as Ref.[11,13,21], with the results:

$$\Delta_{ew}[\bar{\psi}\psi] = 2 + \frac{128}{3N\pi^2},\quad \Delta_{ew}[\bar{\psi}\mathcal{T}_{ew}^A\psi] = 2 - \frac{64}{3N\pi^2}. \quad (8)$$

Here $\mathcal{T}_{ew}^A$ is the generator of the SU($2N$) flavor symmetry. SU($2N$) current operators $\bar{\psi}\gamma_\mu\mathcal{T}_{ew}^A\psi$ gain no anomalous dimension from gauge fluctuations. The order parameters of many competing orders are classified as fermion bilinears of this spin liquid states. For instance, the three sublattice SU(N) columnar valence bond solid order parameter is $\bar{\psi}\mu^x\psi$ plus two other degenerate configurations after translation along x direction. The SU(N) ferromagnetic and antiferromagnet order parameter are $\bar{\psi}\gamma_0 T^a\psi$ and $\bar{\psi}\mu^z T^a\psi$, with $a = 1\cdots N^2-1$. We can see that the VBS and the AF order parameters have the same scaling dimension, and it is smaller than the scaling dimension of FM order parameter based on the $1/N$ expansion. When the four fermion interaction is strong enough there is a transition towards a phase characterized by one of the fermion bilinear order parameters.

We took $J_z = 0$ at the beginning of this section, but the algebraic spin liquid discussed here is stable against small J_z, because J_z will not introduce any gauge invariant relevant perturbation to the field theory Eq. 6. For instance, all the fermion bilinears are ruled out by gauge symmetries and symmetries in Eq. 7 already. Therefore a small J_z only renormalizes four fermion terms to Lagrangian Eq. 6. More physically speaking, turning on small J_z will not allow any orbital triplon state in the low energy Hilbert space, therefore the U(1) $\times$ SU(2) gauge field formalism is still applicable.

For the same reason as the previous paragraph, if we introduce a small perturbation that breaks the orbital Z_2 symmetry exchanging the two orbital levels,

no extra relevant gauge invariant perturbations on Eq. 6 can be induced. This is simply because that the spinon ψ does not carry any physical orbital charge, therefore a small Z_2 symmetry breaking will not be reflected in the CFT. For instance, in the semimetal phase the Z_2 symmetry breaking will lead to a velocity anisotropy between two orbitals: $\sum_{<i,j>} \delta t c_i^\dagger \sigma^z c_j$. Written in terms of fractionalized quantities, this term reads $\sum_{<i,j>} \delta t b_i^\dagger b_j f_{i\alpha}^\dagger h_{i\alpha\mu}^\dagger \sigma_{\mu\nu}^z h_{i\nu\beta} f_{i\beta}$, which breaks the global $\mathrm{SU}(2)_o = \mathrm{SU}(2)_L$ symmetry but still preserves the SU(2) gauge symmetry. The linear order effect of the δt term on the spinon band structure is proportional to $\langle b_i^\dagger b_j h_{i\alpha\mu}^\dagger \sigma_{\mu\nu}^z h_{i\nu\beta}\rangle$, and this expectation value is evaluated in the spin liquid state. Since the spin liquid state is invariant under the Z_2 symmetry $e^{i\frac{\pi}{2}\sigma^x}$, $\langle b_i^\dagger b_j h_{i\alpha\mu}^\dagger \sigma_{\mu\nu}^z h_{i\nu\beta}\rangle = 0$, hence at the linear order the band structure of f is unchanged. In fact, the velocity anisotropy $\sum_{<i,j>} \delta t c_i^\dagger \sigma^z c_j$ can lead to the following gauge invariant but "Lorentz symmetry" breaking coupling in addition to the field theory Eq. 6:

$$\delta L = \sum_{m=1}^{3} s\mathrm{Tr}[h^\dagger \sigma^z h \sigma^m] \bar{\psi} \sigma^m \gamma_k (\partial_k - i a_k - \sum_{l=1}^{3} i A_k^l \frac{\sigma^l}{2}) \psi, \quad (9)$$

where k is x or y. $\mathrm{Tr}[h^\dagger \sigma^z h \sigma^m]$ is odd under the orbital Z_2 transformation $\sigma^z \to -\sigma^z$, or $\exp(i\frac{\pi}{2}\sigma^x)$. Therefore as long as the SU(2) slave rotor h remains gapped, this term only induces irrelevant term after integrating out the gapped h. However, as we will see in the next section, after the condensation of h, an anisotropic velocity of the spin liquid will be induced, and we have to evaluate this anisotropy with RG equation.

SU(2) gauge field has been introduced in SU(2) and more generally $\mathrm{Sp}(2N)$ spin systems with single orbital[20,22–24], but there the local SU(2) gauge symmetry is a transformation mixing particle and holes of spinons, and hence there is no extra U(1) gauge field as in Eq. 6. This particle-hole SU(2) gauge symmetry has no straightforward generalization to larger nonabelian gauge symmetries. In our case the SU(2) gauge field stems from the physical orbital degeneracy, and a straightforward generalization to $\mathrm{SU}(k)$ gauge field with $k-$orbitals can be made, as long as the Hamiltonian favors a total antisymmetric orbital state. In this case we can again decompose $c_{i\alpha m}$ as $c_{i\alpha m} = b_i h_{\alpha\beta} f_{i\beta m}$ with $h \in \mathrm{SU}(k)$. When k is large the $\mathrm{SU}(k)$ gauge field tends to confine gauge charges, and controlled calculations are difficult. However, here $\mathrm{SU}(k)$ gauge field fluctuation corresponds to the constraint of antisymmetric orbital state, which is analogous to large$-S$ of SU(2) spin system with antisymmetric orbitals. Therefore the gauge confined phase could be a semiclassical spin ordered phase.

The credibility of the $\mathrm{U}(1)\times\mathrm{SU}(k)$ gauge field formalism can be tested in one dimension, where many results can be obtained exactly. For instance, one of the fixed points of $k-$orbital $\mathrm{SU}(N)$ spin chain is described by the Wess-Zumino (WZ) model of $\mathrm{SU}(N)$ group at level k[25]. At the $\mathrm{SU}(N)_k$ fixed point, the exact scaling dimension of the Neel order parameter is $\Delta = \frac{N^2-1}{N(N+k)}$. If we apply the $\mathrm{U}(1)\times\mathrm{SU}(k)$ gauge field formalism to this spin chain, the first order $1/N$ expansion gives the scaling dimension of Neel order $\Delta = 1 - \frac{k}{N}$, which is consistent with the exact result. The equivalence between WZ theory and the constrained fermion was proved in Ref.[26]. In one dimensional spin chains, the WZ fixed point is usually not stable[27] with half-filling, in the $\mathrm{U}(1)\times\mathrm{SU}(k)$ gauge field formalism this instability is due to the relevant Umklapp four-fermion terms for arbitrarily large N. However, in 2+1d all the four-fermion interactions are irrelevant with large enough N, therefore at the field theory level the spin liquid is more realistic in 2+1d than 1+1d.

Recently it was proposed that the most general ground state for $\mathrm{SU}(N)$ Heisenberg magnet with fundamental representation on each site is a gapped chiral spin liquid[28]. A chiral spin liquid can be obtained by spontaneously developing time-reversal and reflection symmetry breaking fermion gap $\bar{\psi}\psi$ in the $\mathrm{U}(1)\times\mathrm{SU}(2)$ algebraic spin liquid, which will lead to the Chern-Simons topological field theory for the gauge fields.

The $\mathrm{U}(1)\times\mathrm{SU}(2)$ spin liquid state is very constrained, since both the half-filling constraint and $\mathrm{SU}(2)_o$ singlet constraint are imposed on each site of the lattice. In the following we will study several other liquid states in the same system, which can be obtained from softening part of the constraints on the $\mathrm{U}(1)\times\mathrm{SU}(2)$ spin liquid state. Therefore the $\mathrm{U}(1)\times\mathrm{SU}(2)$ spin liquid state is the "mother" state of everything else in this paper.

B. U(1) spin-orbital liquid

Now let us take U large, while keeping J and J_z small. When U becomes dominant, the system forbids charge fluctuations, but allows for coupled spin and orbital fluctuations. In this case we can just introduce chargeon b_i and spinon $f_{i\alpha m}^{(1)}$ as $c_{i\alpha m} = b_i f_{i\alpha m}^{(1)}$ with a local U(1) gauge symmetry. Here the spinon $f_{i\alpha m}^{(1)}$ is equivalent to $\sum_\beta h_{i\alpha\beta} f_{i\beta m}$ with $f_{i\beta m}$ defined in the previous section. Therefore the new spinon does not carry any SU(2) gauge charge, but carries the physical $\mathrm{SU}(2)_o$ charge. If the fermionic spinon $f_{i\alpha m}$ fills the same mean field band structure as the original fermions $c_{i\alpha m}$ *i.e.* $\langle b_i^\dagger b_j\rangle$ is a constant, the low energy field theory of this spin-orbital liquid is described by the following 3D QED Lagrangian:

$$\mathcal{L}_{qed} = \sum_{a=1}^{4N} \bar{\psi}_a \gamma_\mu (\partial_\mu - i a_\mu) \psi_a + \cdots \quad (10)$$

with global flavor symmetry $\mathrm{SU}(4N)$. The existence of this phase has been shown with a U(1) slave rotor mean-field calculation in Ref.[19]. This type of Lagrangian has been studied quite extensively in the past, because several other spin liquid states also have the 3D QED as their low energy effective field theory[11,13,21]. It is well-known that when $N_f = 4N$ is larger than a critical number,

the 3D QED is a CFT[29]. Since this CFT fixed point involves both spin and orbital degrees of freedom (but no charge fluctuation), we will call this CFT fixed point a U(1) spin-orbital liquid.

In the well-known staggered flux state of SU(2) spin system, $N_f = 4$[11,21,24], while in the spin-orbital liquid states of alkaline earth atoms under study, $N_f = 4N$ can be as large as 40, therefore it is a much more promising system to realize this CFT. The first order $1/N_f$ expansion gives us the following results for the scaling dimensions:

$$\Delta_{qed}[\bar{\psi}\psi] = 2 + \frac{32}{3N\pi^2}, \quad \Delta_{qed}[\bar{\psi}\mathcal{T}^A_{qed}\psi] = 2 - \frac{16}{3N\pi^2} \quad (11)$$

$\mathcal{T}^A_{qed}$ is the generator of the SU(N_f) flavor symmetry group. Again the SU(N_f) current $\bar{\psi}\gamma_\mu \mathcal{T}^A_{qed}\psi$ gains zero anomalous dimension. We can compare the U(1) gauge field formalism and $1/N$ expansion to the exact result of SU($2N$) chains in one dimension, and the $1/N$ expansion gives the exact result as WZ model at level $k = 1$.

Now let us again introduce the Z_2 symmetry breaking perturbation $\sum_{<i,j>} \delta t c_i^\dagger \sigma^z c_j = \sum_{<i,j>} \delta t b_i^\dagger b_j f_i^{(1)\dagger}\sigma^z f_j^{(1)}$. Unlike the U(1) × SU(2) spin liquid discussed in the previous section, since $\langle b_i^\dagger b_j\rangle \neq 0$, now this Z_2 symmetry breaking will introduce the following gauge invariant perturbation to the field theory Eq. 10, which cannot be absorbed by rescaling ψ:

$$\delta L = s\bar{\psi}\sigma^3\gamma_k(\partial_k - ia_k)\psi, \quad (12)$$

here $k = x, y$ only includes the spatial coordinates. Physically this term corresponds to the velocity difference between the e and g orbitals, and it can be viewed as Eq. 9 after the condensation of SU(2) slave rotor h. We can evaluate the RG flow of this term at the order of $1/N$ trough Feynman diagrams Fig. 1b, c and d, as was done in Ref.[11]. And the result is that

$$\frac{ds}{d\ln l} = -\frac{16}{15\pi^2 N}s. \quad (13)$$

Therefore this perturbation is irrelevant under RG flow.

Now if we gradually increase J in Eq. 1, finally the orbital triplons will be excluded from the low energy Hilbert space, and the U(1) × SU(2) spin liquid state discussed in the previous section becomes the candidate ground state. The phase transition between the U(1)×SU(2) spin liquid and the U(1) spin-orbital liquid can be driven by condensing the triplon field $h_{\alpha\beta}$, which can also be parametrized as $h = \phi_0 I + i\phi_1\sigma^1 + i\phi_2\sigma^2 + i\phi_3\sigma^3$, $\vec{\phi}$ is a real O(4) vector, and σ^a are Pauli matrices. Further we can define CP(1) field $z = (z_1, z_2)^t$, and $z_1 = \phi_0 - i\phi_3$, $z_2 = \phi_2 - i\phi_1$. Now this phase transition can be described by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{ew} + |(\partial_\mu - \sum_{l=1}^{3} iA^l_\mu \frac{\sigma^l}{2})z|^2 + r_z|z|^2 + \cdots \quad (14)$$

with critical point $r = 0$. $\mathcal{L}_{ew}$ is given by Eq. 6. After the condensation of z, all three SU(2) gauge field A^l_μ will be higgsed and gapped out, and the remnant gauge field is a_μ. Based on the definition of f in Eq. 3, the condensation of $h_{\alpha\beta}$ implies that $f^{(1)}$ and f becomes equivalent after a global SU(2) rotation. This phase transition is beyond the Landau's theory, because neither side of the phase transition can be characterized by an order parameter. For general SU(k) gauge symmetry with $k > 2$, condensation of matrix field $h_{\alpha\beta}$ always gaps out all components of nonabelian gauge fields.

Since the fermion number in $\mathcal{L}_{ew}$ is large, one can use a systematic $1/N$ expansion to study the universality class of the transition Eq. 14, and the large fermion flavor number will suppress the SU(2) gauge fluctuations. For instance, in the large-N limit, we can view the SU(2) gauge field completely suppressed, then the transition described by Eq. 14 belongs to the 3d O(4) universality class. Notice that other gauge invariant and symmetry allowed couplings between z and ψ are at very high order, and hence are irrelevant at this transition. For instance, coupling $|z|^2\bar{\psi}\gamma_0\psi$ violates the particle-hole symmetry. And the Z_2 symmetry breaking term Eq. 9 is also irrelevant at this transition due to its high scaling dimension from power-counting.

The phase transition between the ordinary Dirac semimetal phase and the U(1) spin-orbital liquid phase can be driven by condensing the chargeon b in Eq. 3 described by Lagrangian

$$\mathcal{L} \sim \mathcal{L}_{qed} + |(\partial_\mu - ia_\mu)b|^2 + r_b|b|^2 + \cdots \quad (15)$$

$\mathcal{L}_{qed}$ is given by Eq. 10. The condensation of chargeon b will higgs the U(1) gauge field a_μ, and release the charge fluctuation from the constrained Hilbert subspace. In the large-N limit when the gauge field fluctuation is frozen by fermions, Eq. 15 is a 3d XY transition. The velocity anisotropy Eq. 12 is an irrelevant perturbation at this transition as well, because the fluctuation of b will not affect the RG equation Eq. 13 at the order of $1/N$. A similar metal and weak Mott insulator transition is studied in Ref.[30–32], where the condensation of the chargeon rotor b kills the U(1) gauge field fluctuation, and drive the Mott insulator with spinon into an ordinary metal.

C. SU(2) spin-charge liquid

The next situation we will consider is to keep J large, and make J_z and U small. In this case the system forbids triplon excitations, but charge excitations are allowed. We can introduce spinon $f^{(2)}$ as $c_{i\alpha m} = h_{i\alpha\beta}f^{(2)}_{i\beta m}$, and $h_{\alpha\beta}$ is the same SU(2) slave rotor as introduced in Eq. 3, while $f^{(2)}_{i\alpha m}$ is equivalent to $b_i f_{i\alpha m}$. This spin-charge liquid state has meanfield solution $\langle h^\dagger_{i\alpha\rho}h_{j\rho\beta}\rangle \sim \delta_{\alpha\beta}$, and at low energy can be described by Dirac fermions coupled

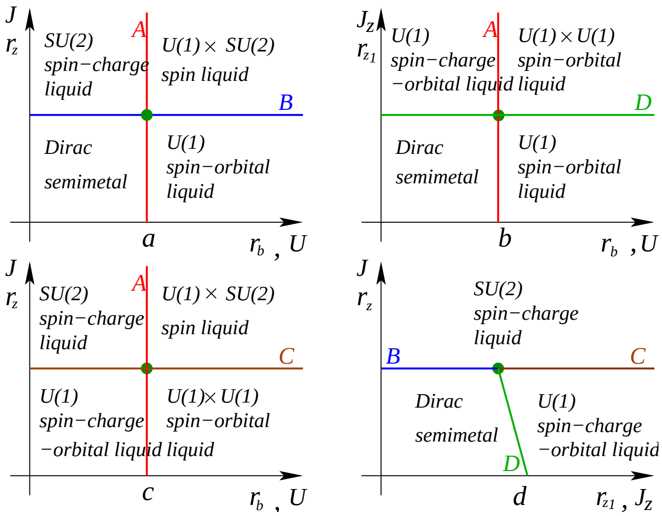

FIG. 2: Schematic phase diagrams with two tuning parameters. All the phase transitions denoted as A are described as a Higgs transition of U(1) slave rotor b, such as Eq. 15 and Eq. 18; phase transitions denoted as B are Higgs transition of SU(2) slave rotor $h_{\alpha\beta}$, or CP(1) field z coupled with SU(2) gauge field, such as Eq. 14 and Eq. 19; phase transitions denoted as C are Higgs transition of SU(2) vector $\vec{\chi}$ coupled with SU(2) gauge field, for example Eq. 23. The phase transitions D are Higgs transition of spinon z_1 coupled with gauge field A^3_μ. Notice that in (a), J_z is weak, while in (c) J_z is strong and the constraint $T^z = 0$ is always imposed on each site. The multi-critical points in these phase diagrams are discussed in section III.

with only SU(2) gauge field with a QCD like Lagrangian

$$\mathcal{L}_{qcd} = \sum_{a=1}^{2N} \bar{\psi}_a \gamma_\mu (\partial_\mu - \sum_{l=1}^{3} i A^l_\mu \frac{\sigma^l}{2}) \psi_a + \cdots \qquad (16)$$

Since this state involves both spin and charge excitations, we will call it a spin-charge liquid. At first glance, the global symmetry in Eq. 16 is SU($2N$) × U(1), but the true symmetry is actually Sp($4N$) ⊃ SU($2N$)×U(1), and this Sp($4N$) group is a subgroup of the O($8N$) symmetry group of the Dirac fermions in the semimetal phase without coupling to any gauge field. The enlarged Sp($4N$) symmetry was discussed in Ref.[12] in the $\pi-$flux state of Sp($2N$) magnets with the same field theory as Eq. 16.

The Sp($4N$) symmetry not only contains the explicit SU($2N$) flavor symmetry of Eq. 16, but also involves the pairing channel of ψ, because now there is no U(1) gauge field, and the gauge singlet cooper pairs of ψ are physical operators. When $k = 2$, The physical order parameters have scaling dimensions

$$\Delta_{qcd}[\bar{\psi}\psi] = 2 + \frac{32}{N\pi^2}, \quad \Delta_{qcd}[\bar{\psi}\mathcal{T}^A\psi] = 2 - \frac{16}{N\pi^2}. \qquad (17)$$

$\mathcal{T}^A \in$ SU($2N$), and there are other fermion pairing bilinears with the same scaling dimension as $\bar{\psi}\mathcal{T}^A\psi$ due to the enlarged Sp($4N$) symmetry. The enlarged symmetry is special for $k = 2$, for QCD Lagrangian with SU(k) gauge group with $k > 2$, since it is impossible to form SU(k) singlet cooper pair, the global symmetry is the apparent SU($2N$) × U(1) symmetry.

Again the SU(2) spin-charge liquid can be obtained from the U(1)×SU(2) spin liquid by "releasing" the charge degree of freedom, through condensing chargeon field b_i in Eq. 3. The Lagrangian is similar to Eq. 15:

$$\mathcal{L} \sim \mathcal{L}_{ew} + |(\partial_\mu - i a_\mu) b|^2 + r_b |b|^2 + \cdots \qquad (18)$$

After the condensation of b, f and $f^{(2)}$ are identical based on their definitions. The transition between the SU(2) spin-charge liquid and the ordinary semimetal phase can be described by condensing triplon z_α from the SU(2) spin-charge liquid state, with Lagrangian similar to Eq. 14:

$$\mathcal{L} \sim \mathcal{L}_{qcd} + |(\partial_\mu - \sum_{l=1}^{3} i A^l_\mu \frac{\sigma^l}{2}) z|^2 + r_z |z|^2 + \cdots \qquad (19)$$

After the condensation of the CP(1) field, physical fermion $c_{i\alpha m}$ and spinon $f^{(2)}_{i\alpha m}$ are identical after a global SU(2) rotation. In the large-N limit, Eq. 18 and Eq. 19 describe a $3d$ XY and $3d$ O(4) transition respectively. A similar orbital Z_2 breaking term is present in the field theory Eq. 16 and Eq. 19, but again this term only leads to irrelevant effects.

D. U(1) × U(1) spin-orbital liquid

If J is small compared with t, while both U and J_z are much larger, then although the charge fluctuation will still be forbidden, the Hamiltonian gives a green light to one component of the orbital triplet state: the state $(|e,g\rangle + |g,e\rangle)/\sqrt{2}$ with $T^z = 0$. Therefore there are two U(1) constraints on the system: $n_e + n_g = N$, $n_e - n_g = 0$, therefore we need to introduce spinon which is invariant under both U(1) charge rotation and orbital rotation generated with T^z. Therefore in the proximity of the semimetal phase, the most natural liquid state with these constraints on the honeycomb lattice is described by the Lagrangian with two U(1) gauge fields

$$\mathcal{L}_{qed2} = \sum_{a=1}^{2N} \bar{\psi}_a \gamma_\mu (\partial_\mu - i a_\mu - i A^3_\mu \frac{\sigma^3}{2}) \psi_a + \cdots \qquad (20)$$

with flavor symmetry SU($2N$)$_+$×SU($2N$)$_-$×Z_2. The two SU($2N$)$_\pm$ groups are generated by $\mathcal{T}^A_\pm = \mathcal{T}^A_{ew}(1 \pm \sigma^3)/2$ respectively, and the Z_2 symmetry exchanges ±. The scaling dimensions of gauge invariant operators to the first order of $1/N$ are

$$\Delta_{qed2}[\bar{\psi}\psi] = 2 + \frac{64}{3N\pi^2}, \quad \Delta_{qed2}[\bar{\psi}\mathcal{T}^A_\pm\psi] = 2 - \frac{32}{3N\pi^2} \qquad (21)$$

If we turn on the Z_2 symmetry breaking perturbation on the lattice, as in the third section, the following term will

be induced in the field theory:

$$\delta L = s\bar{\psi}\sigma^3\gamma_k(\partial_k - ia_k)\psi - \frac{s}{2}iA^3_\mu\bar{\psi}\gamma_k\psi. \quad (22)$$

The scaling dimension of this perturbation can be calculated in the same way as the third section, and it is still irrelevant according to the RG equation at the order of $1/N$. In the rest of this paper this Z_2 symmetry breaking will not be mentioned unless it is relevant.

Since the U(1)×U(1) spin-orbital liquid only allows one of the orbital triplet states, We can obtain the U(1) × U(1) spin-orbital liquid by higgsing two components of the SU(2) gauge field in the U(1) × SU(2) spin liquid discussed before. We already showed that condensing a fundamental spinor of the SU(2) gauge group will gap out all three components of the gauge fields, but if we just condense an adjoint vector of SU(2) gauge group, only two of the three components of the gauge field will be gapped. Therefore starting with the "mother" state U(1) × SU(2) spin liquid, the U(1) × U(1) spin-orbital liquid can be obtained by condensing SU(2) vector $\vec{\chi} = z^\dagger\sigma^a z$ instead of z itself. This transition can be described by the field theory

$$\mathcal{L} = \mathcal{L}_{ew} + \sum_{i=1}^{3}\frac{1}{g}(\partial_\mu\chi_i - \sum_{j,k=1}^{3}\epsilon_{ijk}A^j_\mu\chi_k)^2 + \cdots \quad (23)$$

ϵ_{ijk} is the total antisymmetric tensor, and also the adjoint representation of SU(2) gauge group: $t^a_{ij} = i\epsilon_{aij}$. Without loss of generality, we take $\vec{\chi}$ condense along the direction $(0, 0, 1)$, then A^1_μ and A^2_μ are gapped out, while A^3_μ remains gapless, which is the same as the U(1)×U(1) spin-orbital liquid. Notice that $\vec{\chi}$ is not a vector of the physical $\mathrm{SU}(2)_L$ symmetry. To see this explicitly, we can rewrite $\vec{\chi}$ as

$$\vec{\chi} = z^\dagger\vec{\sigma}z \sim \mathrm{Tr}[h^\dagger\sigma^z h\vec{\sigma}]. \quad (24)$$

h is the SU(2) rotor introduced in Eq. 3. It is explicit in this equation that $\vec{\chi}$ is only invariant under the U(1) subgroup generated by T^z, which is the physical symmetry of the system with finite J_z. Similarly, if we condense the SU(2) vector $\vec{\chi}_1 \sim \mathrm{Tr}[h^\dagger\sigma^x h\vec{\sigma}]$ from the mother state, we would obtain a state with constraint $T^x = 0$ on each site.

In the large-N limit the SU(2) gauge field is again suppressed by the fermions, and the transition Eq. 23 is a $3d$ O(3) transition. A similar phase transition was discussed in a different context[20]. This field theory was also used as a trial unified theory of electro-weak interaction in particle physics, and the gapless A^3_μ was identified as the photon[33]. However, nature chooses a different theory. For larger k, condensing adjoint vector of SU(k) gauge group always leaves some components of the gauge field gapless.

E. U(1) spin-charge-orbital liquid

Finally, if we only keep J_z large, while keeping U and J both small, the only constraint on the system is $T^z_j = 0$ on each site. Then the candidate liquid state in this case is described by the following lagrangian

$$L_{qed3} = \sum_{a=1}^{2N}\bar{\psi}_a\gamma_\mu(\partial_\mu - iA^3_\mu\frac{\sigma^3}{2})\psi_a + \cdots \quad (25)$$

This state has spin, charge and two orbital states fluctuation, therefore following our convention this state will be called U(1) spin-charge-orbital liquid. This state can be obtained from the U(1)×U(1) spin-orbital liquid state by condensing chargeon b. Moreover, this U(1) spin-charge liquid state can also be obtained from condensing SU(2) gauge vector $\vec{\chi}$ in the SU(2) spin-charge liquid Eq. 16, which gaps out both A^1_μ and A^2_μ. In the large-N limit, these two transitions belong to the $3d$ XY and $3d$ O(3) universality class respectively.

We can also drive a direct transition between the U(1) spin-charge-orbital liquid and the semimetal phase, as long as we can gap out the U(1) gauge field A^3_μ in Eq. 25. In the previous paragraph we mentioned that the U(1) spin-charge-orbital liquid state can be obtained from condensing vector $\vec{\chi}$ in the SU(2) spin-charge liquid Eq. 16. After the condensation of $\vec{\chi}$, the degeneracy between the two CP(1) fields z_1 and z_2 is lifted, due to the gauge invariance coupling $\vec{\chi}\cdot z^\dagger\vec{\sigma}z$. Therefore z_1 and z_2 can condense separately, but not together. If one of z_a condenses, it will Higgs the gauge field A^3_μ, and drive the system into the ordinary semimetal phase. This transition can be described by the field theory

$$\mathcal{L} = \mathcal{L}_{qed3} + |(\partial_\mu - iA^3_\mu)z_1|^2 + r_{z1}|z_1|^2 + \cdots \quad (26)$$

In the large-N limit this transition again belongs to the $3d$ XY universality class.

III. PHASE DIAGRAM AND MULTI-CRITICAL POINTS

The phase diagram with two tuning parameters J and U and weak constant J_z is depicted in Fig. 2a, with four different liquid phases. And there is a multi-critical point with both masses of z_α and b vanish. At this multi-critical point, the field theory reads

$$\mathcal{L} \sim \mathcal{L}_{ew} + |(\partial_\mu - ia_\mu)b|^2 + |(\partial_\mu - \sum_{l=1}^{3}iA^l_\mu\frac{\sigma^l}{2})z|^2 + \cdot (27)$$

In the large-N limit z and b behave like a $3d$ O(4) and $3d$ XY transition respectively. On top of this field theory, the symmetry allows the interaction between b and z_α, such as $|z|^2|b|^2$. It is well-known that at the $3d$ O(4) and XY transitions, the scaling dimensions $\Delta[|z|^2] > \Delta[|b|^2] > 3/2$[34], therefore this term $|z|^2|b|^2$ is an irrelevant perturbation in the field theory Eq. 27. Notice that in principle the velocity of b and z are different, and the velocities will flow under RG equation with finite N.

Fig. 2c is the phase diagram with two tuning parameters J and U and strong constant J_z, where the orbital

constraint $T^z = 0$ is always imposed. Again there is a multi-critical point with both masses of z_α and b vanish. The field theory at this multi-critical point is

$$\mathcal{L} = \mathcal{L}_{qcd} + \sum_{i=1}^{3} \frac{1}{g}(\partial_\mu \chi_i - \sum_{j,k=1}^{3} \epsilon_{ijk} A^j_\mu \chi_k)^2 + |(\partial_\mu - i a_\mu) b|^2 + \cdots \quad (28)$$

and it is clear that the coupling $|\vec{\chi}|^2|b|^2$ is irrelevant in the large-N limit due to the fact that $\Delta[|\vec{\chi}|^2] > \Delta[|b|^2] > 3/2$.

The multi-critical point in Fig. 2b with tuning parameters J_z and U can be simply described by field theory $|(\partial_\mu - i a_\mu)b|^2 + |(\partial_\mu - A^3_\mu) z_1|^2$, and it is stable against interactions between b and z_1. However, the multi-critical point in Fig. 2d is no longer a simple combination between Eq. 14, Eq. 19 and Eq. 26, because the symmetry of the system allows the coupling $\vec{\chi} \cdot z^\dagger \vec{\sigma} z$. The fate of this relevant perturbation is unclear at this point.

IV. SUMMARY AND EXTENSIONS

In summary, we studied five examples of liquid states motivated by the orbital flavor and large spin symmetry of alkaline earth cold atoms. The schematic phase diagrams are depicted in Fig. 2. Experimentally the spin correlation calculated in this paper can in principle be measured using the momentum density distribution and noise correlation between atom spins proposed in Ref.[35], after releasing the atoms from the trap. The VBS order which breaks the lattice translation symmetry also has algebraic correlation in the liquid states. The two atoms within one valence bond have stronger AF correlation $J \sim t^2/U$ compared with other atoms, and hence tend to move closer to each other from the minima of the wells. This super-lattice structure can also be measured by the density correlation between atoms, which can be detected by the noise correlation[36].

It would also be interesting to test the results of this work by numerically simulating model Eq. 1 as in Ref.[15], since the system is fixed at half-filling, the sign problem of ordinary interacting fermion system is no longer a concern. Analytically it is useful to pursue a slave rotor meanfield calculation like Ref.[19]. The liquid states discussed in our paper is expected to occur in a finite region close to the Mott transition, and by tuning U still larger there can be a transition from our states into a state with background nonzero gauge flux through plaquette, or dimerized valence bond solids[19]. In our case the interplay between the U(1) and SU(2) slave rotors make the meanfield calculation more complicated, and more meanfield variational parameters need to be introduced. We will study this calculation in new future.

In the current work, the universality class of all the phase transitions between different liquid states was only discussed in the large-N limit. Since the number of boson field at this transition is not large, the $1/N$ expansion at the transition is actually nontrivial. Let us take the simplest quantum critical theory Eq. 15 as an example. If there is no gauge field a_μ, the transition is a $3d$ XY transition, whose critical exponents can be obtained by summing over the Feynman diagrams to all orders of an $\epsilon = 4 - d$ expansion. The $1/N$ correction from the gauge field propagator will enter the Feynman diagram at every order of ϵ expansion, therefore it is a nontrivial task trying to sum over all the diagrams at $1/N$ order. However, if we generalize the boson number to large N_b, then a systematic expansion of both $1/N$ and $1/N_b$ can be straightforwardly carried out, as was studied in Ref.[37].

Our formalism can be applied to other multi-orbital magnets, including transition metal oxides with orbital degeneracy. We will explore this possibility in future.

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